A.C.I.M FOR RANDOM INTERMITTENT MAPS : EXISTENCE, UNIQUENESS AND STOCHASTIC STABILITY

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Abstract. We study a random map T which consists of intermittent maps $\{T_k\}_{k=1}^K$ and probability distribution $\{p_{k,\varepsilon}(x)\}_{k=1}^K$. We prove existence of a unique absolutely continuous invariant measure (ACIM) for the random map T. Moreover, we show that, as ε goes to zero, the invariant density of the random system T converges in the L^1-norm to the invariant density of the deterministic intermittent map T_1. The outcome of this paper contains a first result on stochastic stability of intermittent maps.

1. INTRODUCTION

Expanding maps of the interval which admit an indifferent fixed point at 0 are good testing tools for physical systems with intermittent behaviour. Pianigiani proved the existence of ACIM for a certain class of intermittent maps of the interval in [11]. Later, polynomial decay of correlations was proved for such systems independently in ([9, 12]). More recently, Hu and Vaienti generalized these results to general higher dimensional systems [8].

We are interested in perturbations of intermittent maps. In particular when the indifferent fixed point persists under perturbations. Results on statistical stability of intermittent maps with perturbations of this type were obtained in [1, 2]. However, there are no results on the stochastic stability of intermittent maps when the indifferent fixed point persists under perturbations.

In this paper, we study a random map T which consists of two intermittent maps $\{T_k\}_{k=1}^K$ and probability distribution $\{p_{k,\varepsilon}(x)\}_{k=1}^K$. We prove existence of a unique ACIM for the random map T. Moreover, we show that, as ε goes to zero, the invariant density of the random system T converges in the L^1-norm to the invariant density of T_1. We prove our results by using a cone technique. This cone was also used in [10] to study Ulam approximations for deterministic intermittent map.

In section 2, we present the setup of the problem. Section 3 contains the proof of the existence and uniqueness of the ACIM for the random map. Section 4 contains examples of random maps which satisfy our conditions. Section 5 contains the stochastic stability result. In section 6, we show that our random maps give rise to an interesting family of 2-dimensional non-uniformly expanding maps which admit a unique ACIM.

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2. PRELIMINARIES

2.1. **Setup.** Let $(I, \mathcal{B}(I), m)$ be a measure space, where $I = [0,1], \mathcal{B}(I)$ is Borel $\sigma-$algebra and m is Lebesgue measure. For $0 < \beta < \alpha < 1$, let

$$T_1 = \begin{cases} x(1+2^\alpha x^\alpha) & x \in [0, \frac{1}{2}), \\ g_1(x) & x \in [\frac{1}{2}, 1]. \end{cases} \qquad T_2 = \begin{cases} x(1+2^\beta x^\beta) & x \in [0, \frac{1}{2}), \\ g_2(x) & x \in [\frac{1}{2}, 1]. \end{cases}$$

We assume:

(1) $g_k(\frac{1}{2}) = 0, \quad k = 1, 2$;

(2) $g_k^{'}(x) > 1, \quad k = 1, 2.$

Remark 2.1. To simplify the notation in the proofs, we consider that T consists of two maps T_1, T_2. The proofs for any finite $K > 2$ is similar.

We study a position dependent random map

$$T = \{T_1(x), T_2(x); p_1(x), p_2(x)\}$$

which is understood as a Markov process with transition function

$$\mathbb{P}(x, A) = p_1(x)\chi_A(T_1(x)) + p_2(x)\chi_A(T_2(x)),$$

where A is any measurable set in $\mathcal{B}(I)$, and $\{p_1(x), p_2(x)\}$ is a set of position dependent probabilities[1], i.e. $p_1(x) + p_2(x) = 1$ and $0 \le p_1(x), p_2(x) \le 1$, for any $x \in I$, and χ_A is the characteristic function of the set A.

The transition function $\mathbb{P}(x, A)$ induces an operator E_T on measures on $(I, \mathcal{B}(I))$ denoted by

$$\begin{aligned} E_T\mu(A) &= \int_I \mathbb{P}(x, A) d\mu(x) \\ &= \int_I p_1(x)\chi_A(T_1(x)) + p_2(x)\chi_A(T_2(x)) d\mu(x) \\ &= \int_{T_1^{-1}(A)} p_1(x) d\mu(x) + \int_{T_2^{-1}(A)} p_2(x) d\mu(x). \end{aligned}$$

We say that μ is $T-$invariant if and only if

$$E_T\mu(A) = \mu(A);$$

that is, for any measurable set A,

$$\mu(A) = \int_{T_1^{-1}(A)} p_1(x) d\mu(x) + \int_{T_2^{-1}(A)} p_2(x) d\mu(x).$$

If μ has density function f with respect to m, then $E_T\mu$ has also a density function which is defined by $\mathcal{L}_T f$. We obtain following by change of variables, $d\mu = f dm$, for any measurable set A,

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[1] This implies that the selection process of T_1, T_2 is not necessarily an iid process.

$$\begin{aligned}
\int\limits_A \mathcal{L}_T f dm(x) &= E_T\mu(A) = \int\limits_{T_1^{-1}(A)} p_1(x)d\mu(x) + \int\limits_{T_2^{-1}(A)} p_2(x)d\mu(x) \\
&= \int\limits_{T_1^{-1}(A)} p_1(x)fdm(x) + \int\limits_{T_2^{-1}(A)} p_2(x)fdm(x) \\
&= \int\limits_A p_1(T_1^{-1}(x))f(T_1^{-1}(x))dm(x) + \int\limits_A p_2(T_2^{-1}(x))f(T_2^{-1}(x))dm(x) \\
&= \int\limits_A P_{T_1}(p_1 f)dm(x) + \int\limits_A P_{T_2}(p_2 f)dm(x) \\
&= \int\limits_A [P_{T_1}(p_1 f) + P_{T_2}(p_2 f)]dm(x), \qquad (2.1)
\end{aligned}$$

where P_{T_1} and P_{T_2} are Perron-Frobenius operators [6] associated with T_1 and T_2 respectively. Since (2.1) holds for any measurable set A, we will get an almost everywhere equality: [2]

$$\begin{aligned}
(\mathcal{L}_T f)(x) &= P_{T_1}(p_1 f)(x) + P_{T_2}(p_2 f)(x) \\
&= \sum_{y \in T_1^{-1}(x)} \frac{(p_1 f)(y)}{|T_1'(y)|} + \sum_{y \in T_2^{-1}(x)} \frac{(p_2 f)(y)}{|T_2'(y)|}.
\end{aligned}$$

2.2. **Properties.** We call $\mathcal{L}_T$ the Perron-Frobenius operator associated with the random map T and the operator has very useful properties. The properties of $\mathcal{L}_T$ resemble the properties of the classical Perron-Frobenius operator associated with a single deterministic map.

Lemma 2.2. *$\mathcal{L}_T$ satisfies the properties as follows:*
(i)(Linearity) $\mathcal{L}_T : L^1 \to L^1$ *is a linear operator.*
(ii)(Positivity) *Let* $f \in L^1$ *and assume* $f \geq 0$, *then* $\mathcal{L}_T f \geq 0$.
(iii)(Preservation of integrals)

$$\int\limits_I \mathcal{L}_T f dm(x) = \int\limits_I f dm(x)$$

(iv)(contraction) *for any* $f \in L^1$,

$$\| \mathcal{L}_T f \|_1 \leq \| f \|_1$$

(v) $\mathcal{L}_T f = f \Leftrightarrow E_T\mu = \mu$, *i.e measure* $\mu = f \cdot m$ *is* $T-$*invariant.*
(vi)(composition)

$$\mathcal{L}_{T\circ R} f = \mathcal{L}_T \circ \mathcal{L}_R f$$

In particular, $\mathcal{L}_{T^n} f = \mathcal{L}_T^n f$.

Proof. See [4] **Lemma 3.1**. □

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[2] Note that since $p_1(x), p_2(x)$ are functions of $x, \mathcal{L}_T$ is not a convex combination of p_1 and p_2.

2.3. **Notations and Assumptions.** Finally, we set notations and assumptions that we need later in the text. For $x \in I, k \in \{1,2\}$ and partition $\mathcal{P} = \{I_1, I_2\}, I_i = [a_{i-1}, a_i], i = 1,2$, we make the following definitions

$$\begin{aligned} T(x) &= T_k(x), \quad \text{with probability } p_k(x) \\ T^n(x) &= T_{k_n} \circ T_{k_{n-1}} \circ \cdots \circ T_{k_1}(x), \text{ with probability} \\ & \quad p_{k_n}(T_{k_{n-1}} \circ \cdots \circ T_{k_1}(x)) \cdot p_{k_{n-1}}(T_{k_{n-2}} \circ \cdots \circ T_{k_1}(x)) \cdot p_{k_1}(x), \quad k_i \in \{1,2\} \\ T_{k,i} &= T_k \mid_{I_i} . \end{aligned}$$

Assumptions:

(A) $\sum_{i=1}^{l} \frac{p_k(T_{k,i}^{-1}(x))}{T_k'(T_{k,i}^{-1}(x))}, 1 \le l \le 2$, is decreasing for all $k = 1,2$.

(B) $\inf_{x \in I} p_k(x) \ge \delta > 0$.

3. EXISTENCE AND UNIQUENESS OF ACIM

Since the random map T gives exactly two pre-images to the given x; we will adopt the convention that these are

$$T_1^{-1}x = \{y_1, z_1\}, \quad y_1 \le \frac{1}{2} \le z_1,$$

$$T_2^{-1}x = \{y_2, z_2\}, \quad y_2 \le \frac{1}{2} \le z_2.$$

$$y_* = \max\{y_1, y_2\} \in [0, \frac{1}{2}], \quad z_* = \max\{z_1, z_2\} \in [\frac{1}{2}, 1].$$

Define $m(f) = \int_0^1 f(x)dm(x)$, where m is Lebesgue measure.

In this section we prove the following theorem:

Theorem 3.1.
(i) The random map T admits a unique ACIM $\mu, d\mu = \rho dm$.
(ii) The invariant density ρ is uniformly bounded below.

We prove the theorem by a series of Lemmas and propositions.

Cone. For $A > 0$, define

$$\mathcal{C}_A = \{f \in L^1 \mid f \ge 0, f \text{ decreasing}, \int_0^x f dm \le A x^{1-\alpha} m(f)\}.$$

We will show that for sufficient large A the cone $\mathcal{C}_A$ is invariant under the action of operator $\mathcal{L}_T$.

Lemma 3.2. *Let $f \in \mathcal{C}_A$. Then, for $x \in (0,1]$,*
(i) $f(x) \le A x^{-\alpha} m(f)$;
(ii) $f(x) \le \frac{1}{x} m(f)$, *and in particular,* $f(x)|_{x \in [\frac{1}{2}, z_*)} \le 2m(f)$;
(iii) $y_1 \ge \frac{x}{2}, y_2 \ge \frac{x}{2}$ *and* $x \ge y_*$;
(iv) $(1-x)^{1-\alpha} \le 1 - (1-\alpha)x$;
(v) $x^{1-\alpha} - y_*^{1-\alpha} \ge \frac{1-\alpha}{2} x$.

Proof. (i) We have

$$xf(x) = \int_0^x f(x)dm(\xi) \leq \int_0^x f(\xi)dm(\xi) \leq Ax^{1-\alpha}m(f).$$

(ii) By $f(x) \geq 0$ and decreasing, we have

$$xf(x) = \int_0^x f(x)dm(\xi) \leq \int_0^x f(\xi)dm(\xi) \leq \int_0^x f(\xi)dm(\xi) + \int_x^1 f(\xi)dm(\xi) = m(f).$$

So, $f(x) \leq \frac{1}{x}m(f)$ and in particular $f(x) \leq 2m(f)$, when $x \in [\frac{1}{2}, z_*)$.

(iii) For $y_1, y_2 \leq \frac{1}{2}, 0 < \beta < \alpha < 1$, we have

$$x = T_1(y_1) = y_1(1 + 2^\alpha y_1^\alpha) \leq 2y_1 \text{ and } x = T_2(y_2) = y_2(1 + 2^\alpha y_2^\beta) \leq 2y_2.$$

Also,

$$x = T_1(y_1) = y_1(1 + 2^\alpha y_1^\alpha) \geq y_1 \text{ and } x = T_2(y_2) = y_2(1 + 2^\alpha y_2^\beta) \geq y_2.$$

Therefore, $y_1 \geq \frac{x}{2}, y_2 \geq \frac{x}{2}$ and $x \geq y_*$.

(iv) Set

$$g(x) = (1-x)^{1-\alpha} - [1 - (1-\alpha)x],$$

then $g(0) = 1 - 1 = 0$ and for $x \in [0, 1]$,

$$g^{'}(x) = -(1-\alpha)(1-x)^{-\alpha} + (1-\alpha) = (1-\alpha)[1 - \frac{1}{(1-x)^\alpha}] \leq 0.$$

So, $g(x) \leq 0, x \in (0, 1]$ can be achieved from above two properties, that is

$$(1-x)^{1-\alpha} \leq 1 - (1-\alpha)x.$$

(v) First write,

$$x^{1-\alpha} - y_*^{1-\alpha} = x^{1-\alpha}[1 - (\frac{y_*}{x})^{1-\alpha}] = x^{1-\alpha}[1 - (1 - \frac{x - y_*}{x})^{1-\alpha}].$$

Let $\zeta = \frac{x - y_*}{x}$.

In case $y_* = y_1$,

$$x = T_1(y_1) = y_1(1 + 2^\alpha y_1^\alpha) > y_1 > 0, \quad x \leq 2y_1 \quad \text{and } \zeta = \frac{x - y_1}{x} \in (0, 1].$$

Thus,

$$\begin{aligned} x^{1-\alpha} - y_*^{1-\alpha} &= x^{1-\alpha}[1 - (1-\zeta)^{1-\alpha}] \\ &\geq x^{1-\alpha}[1 - (1 - (1-\alpha)\zeta)] \\ &= x^{1-\alpha}(1-\alpha)\frac{x - y_1}{x} \\ &= x^{-\alpha}(1-\alpha)(T_1(y_1) - y_1) \\ &= x^{-\alpha}(1-\alpha)(2^\alpha y_1^{\alpha+1}) \\ &\geq (2y_1)^{-\alpha}(1-\alpha)(2^\alpha y_1^{\alpha+1}) \\ &= (1-\alpha)y_1 \\ &\geq \frac{(1-\alpha)}{2}x. \end{aligned}$$

In case $y_* = y_2$,

$$x = T_2(y_2) = y_2(1 + 2^\beta y_2^\beta) > y_2 > 0, \quad x \leq 2y_2 \quad \text{and } \zeta = \frac{x - y_2}{x} \in (0, 1].$$

We have

$$\begin{aligned} x^{1-\alpha} - y_*^{1-\alpha} &= x^{1-\alpha}[1 - (1-\zeta)^{1-\alpha}] \\ &\geq x^{1-\alpha}[1 - (1 - (1-\alpha)\zeta)] \\ &= x^{1-\alpha}(1-\alpha)\frac{x - y_2}{x} \\ &= x^{-\alpha}(1-\alpha)(T_2(y_2) - y_1) \\ &= x^{-\alpha}(1-\alpha)(2^\beta y_2^{\beta+1}) \\ &\geq (2y_2)^{-\alpha}(1-\alpha)(2^\beta y_2^{\beta+1}) \\ &= (1-\alpha)y_2(2y_2)^{\beta-\alpha}. \end{aligned}$$

In our notation, we have $0 < \beta < \alpha < 1$ and $0 \leq 2y_2 \leq 1$, then $(2y_2)^{\beta-\alpha} \geq 1$. Thus,

$$x^{1-\alpha} - y_*^{1-\alpha} \geq (1-\alpha)y_2 \geq (1-\alpha)\frac{x}{2}.$$

$\square$

Lemma 3.3. *Let $f \geq 0$ be a decreasing function. Then $\mathcal{L}_T f$ is also decreasing.*

Proof. See Lemma 3.1 of [5]. [3] $\square$

Proposition 3.4. *For $A \geq \frac{4}{1-\alpha}$ the cone $\mathcal{C}_A$ is invariant under the action of the operator $\mathcal{L}_T$.*

Proof. By Lemma 3.3, for $f \in \mathcal{C}_A$ we know that $\mathcal{L}_T f$ is decreasing . Also, $\mathcal{L}_T f \geq 0$ and $m(\mathcal{L}_T f) = m(f)$. Therefore we only need to prove that

$$\int_0^x \mathcal{L}_T f dm \leq Ax^{1-\alpha} m(\mathcal{L}_T) = Ax^{1-\alpha} m(f),$$

when $A \geq A_* = \frac{4}{1-\alpha}$. We have

$$\begin{aligned} \int_0^x \mathcal{L}_T f dm &= \int_0^x P_{T_1}(p_1 f) + P_{T_2}(p_2 f) dm = \int_{T_1^{-1}[0,x]} (p_1 f) dm + \int_{T_2^{-1}[0,x]} (p_2 f) dm \\ &= (\int_0^{y_1} + \int_{\frac{1}{2}}^{z_1})(p_1 f) dm + (\int_0^{y_2} + \int_{\frac{1}{2}}^{z_2})(p_2 f) dm \\ &\leq (\int_0^{y_*} + \int_{\frac{1}{2}}^{z_*})(p_1 f) dm + (\int_0^{y_*} + \int_{\frac{1}{2}}^{z_*})(p_2 f) dm \\ &= \int_0^{y_*} (p_1 + p_2) f dm + \int_{\frac{1}{2}}^{z_*} (p_1 + p_2) f dm \leq A y_*^{1-\alpha} m(f) + \int_{\frac{1}{2}}^{z_*} f dm, \end{aligned}$$

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[3] Note that this Lemma only requires assumption (A) to hold.

where $y_* = \max\{y_1, y_2\} \in [0, \frac{1}{2}]$, $\quad z_* = \max\{z_1, z_2\} \in [\frac{1}{2}, 1]$. Since our transformations $T_k(x)$ is not onto, there are two cases to talk about.

In the case 1, there is no z_*. That is, we have only one pre-image to the given x. By the Lemma 3.2 (iii), we get $x \geq y_*$. So, $y_*^{1-\alpha} \leq x^{1-\alpha}$, with $1-\alpha > 0$. Therefore, $\int\limits_0^x \mathcal{L}_T f dm \leq A y_*^{1-\alpha}$ for $A > 0$.

In the case 2, z_* exists. From Lemma 3.2, we have $f(x) \leq 2, x \in [\frac{1}{2}, z_*]$. Then,

$$\int\limits_{\frac{1}{2}}^{z_*} f dm \leq \int\limits_{\frac{1}{2}}^{z_*} 2m(f) dm = 2(z_* - \frac{1}{2}) m(f).$$

Moreover, we have $g_1^{'}(x) > 1, g_2^{'}(x) > 1$, then $x = m[0,x] = m \circ T_k[\frac{1}{2}, z_k] \geq z_k - \frac{1}{2}, k = 1, 2$ i.e. $x > z_* - \frac{1}{2}$. So,

$$\int\limits_{\frac{1}{2}}^{z_*} f dm < 2x m(f).$$

By the result of Lemma 3.2(iv), we attain that $x \leq \frac{2}{1-\alpha}(x^{1-\alpha} - y_*^{1-\alpha})$. Then, for $A \geq A_* = \frac{4}{1-\alpha}$,

$$\int\limits_0^x \mathcal{L}_T f dm < A y_*^{1-\alpha} m(f) + \frac{4}{1-\alpha}(x^{1-\alpha} - y_*^{1-\alpha}) m(f) \leq A x^{1-\alpha} m(f).$$

Therefore, $\mathcal{L}_T f \in \mathcal{C}_A$, with $f \in \mathcal{C}_A$ and $A \geq \frac{4}{1-\alpha}$. □

Obviously, if $f \in \mathcal{C}_A$ and $A \geq= \frac{4}{1-\alpha}$, then $\mathcal{L}_T^n f \in \mathcal{C}_A, n \geq 1$.

Remark 3.5. Since $\mathcal{C}_A$ is compact and convex, operator $\mathcal{L}_T$ has a fixed point $f_* \in \mathcal{C}_A$ by Proposition 3.4 and the Schauder-Tychonoff fixed point theorem of [7]. Thus, random map T admits an ACIM .

Let μ be an ACIM for random map T. See Appendix, we know each of maps T_k has a unique ACIM. Then let ν_1 and ν_1 be the unique ACIM for T_1 and T_2 respectively. Define $A_k = \text{supp}(\nu_k)$ and basins $\mathcal{U}_k = \bigcup\limits_{j=0}^{\infty} T_k^{-j} A_k$. By the uniqueness of ACIM for T_k on $I = [0, 1]$, we have

$$A_k = \mathcal{U}_k = I, \quad k = 1, 2.$$

Lemma 3.6. *For $k = 1, 2, I = A_k \subseteq supp(\mu)$.*

Proof. Since $A_k = \mathcal{U}_k = I$ for $k = 1, 2$. Then $\mu(A_k) = \mu(\mathcal{U}_k) > 0$.

Let $B = I \cap \text{supp}(\mu)$, then $B \neq \emptyset$ and $\mu(B) > 0$. Since B is subset of $I = A_k$ and A_k is an invariant set, then $\bigcup\limits_{i=0}^{\infty} T_k^i B \subseteq A_k$.

Assume $A_k \nsubseteq \text{supp}(\mu)$. Then $\mu(A_k \setminus B) = 0$. Also,

$$\mu(A_k \setminus B) \geq \mu(\bigcup_{i=0}^{\infty} T_k^i B \setminus B) = \mu(\bigcup_{i=1}^{\infty} T_k^i B \setminus B) \geq \mu(T_k^i B), \quad i = 1, 2, ...$$

So, in this case, $\mu(T_k^iB) \leq 0, i = 1, 2, ...$ However this leads to a contradiction because under **condition (B)**,

$$\begin{aligned}\mu(T_1B) &= \int\limits_{T_1^{-1}(T_1B)} p_1 d\mu + \int\limits_{T_2^{-1}(T_1B)} p_2 d\mu \\ &\geq \inf_{x\in I} p_1(x)\mu(B) + \inf_{x\in I} p_2(x)\mu(T_2^{-1}(T_1B)) > 0.\end{aligned}$$

and

$$\begin{aligned}\mu(T_2B) &= \int\limits_{T_1^{-1}(T_2B)} p_1 d\mu + \int\limits_{T_2^{-1}(T_2B)} p_2 d\mu \\ &\geq \inf_{x\in I} p_1(x)\mu(T_1^{-1}(T_2B)) + \inf_{x\in I} p_2(x)\mu(B) > 0\end{aligned}$$

Therefore, $I = A_k \subseteq \mathrm{supp}(\mu)$. □

Proposition 3.7. *Let $A \geq A_* = \frac{4}{1-\alpha}$ and $f \in \mathcal{C}_A$. Then for random map T, there are $\gamma > 0, N \in \mathbb{Z}_+$ such that $\mathcal{L}_T^n f \geq \gamma m(f)$, for all $n \geq N$, where γ, N depend only on A. In particular, if also $\rho = \mathcal{L}_T\rho$ then $\mu = \rho m$ is equivalent to m.*

Proof. First we know from Proposition 3.4, if $A \geq \frac{4}{1-\alpha}, f \in \mathcal{C}_A$, then $\mathcal{L}_T^n f \in \mathcal{C}_A$. So, we have

$$\int\limits_0^x f dm \leq Ax^{1-\alpha}m(f), \int\limits_0^x \mathcal{L}_T^n f dm \leq Ax^{1-\alpha}m(\mathcal{L}_T^n f).$$

Without loss of generality, we suppose that $m(f) = 1$ and then $m(\mathcal{L}_T^n f) = m(f) = 1$. Therefore we only need to prove $\mathcal{L}_T^n f \geq \gamma$.

Now, we fix a small number $0 < \sigma < \frac{1}{2}$, such that $A\sigma^{1-\alpha} = \frac{1}{2}$. Then,

$$\int\limits_0^\sigma f dm \leq A\sigma^{1-\alpha} = \frac{1}{2} \text{ and } \int\limits_\sigma^1 f dm = 1 - \int\limits_0^\sigma f dm \geq \frac{1}{2}.$$

When $x \in (0, \sigma)$, we have

$$f(x) \geq f(\sigma) = \frac{\int_\sigma^1 f(\sigma)dm}{1-\sigma} \geq \frac{\int_\sigma^1 f(x)dm}{1-\sigma} \geq \frac{1}{2(1-\sigma)},$$

since $f(x)$ is a decreasing function.

Also, $\mathcal{L}_T^n f(x)$ is decreasing. Then it is enough to show that $\mathcal{L}_T^n f(1)$ is bounded below away from zero. By Lemma 2.1(vi) we get $\mathcal{L}_T^n f(1) = \mathcal{L}_{T^n} f(1)$.

Define $x_n = T^{-1}(x_{n-1}) \cap [0, \frac{1}{2}], n \geq 1$ and $x_0 = 1$ Obviously, $\{x_n\}$ is a strictly decreasing sequence and it converges to 0. Set

$$\omega_n = \{k_1, k_2, ..., k_n \in \{1, 2\}^n\}, k_i \in \{1, 2\}.$$

So, $\{x_n\}$ depends on ω_n. We denote $\{x_{n,\omega_n}\} = \{x_n\}(\omega_n)$.

With the fixed σ, we can find an N such that $\{0, b_1, b_2, ..., b_q\}$ are critical points of map T_{ω_N} and

$$\{b_1, b_2\} = T_{k_1}^{-1}(x_{N-1}), \quad \max_{\omega_N} x_{N-1,\omega_N} \leq \sigma.$$

Then, for all ω_N, we have $\mathcal{L}_T f(x_{N-1,\omega_N}) \geq \mathcal{L}_T f(\sigma)$ since $\mathcal{L}_T f(x)$ is decreasing.

$$\begin{aligned}
\mathcal{L}_{T^N} f(1) &= \sum_{\omega_N \in \{1,2\}^N} \sum_{i=1}^{q} \frac{(p_{\omega_N} f)(b_i)}{T'_{\omega_N}(b_i)} \\
&= \sum_{\omega_N \in \{1,2\}^N} \sum_{i=1}^{q} \frac{p_{\omega_{N-1}}(T_{k_1}(b_i)) p_{k_1}(b_i) f(b_i)}{T'_{\omega_{N-1}}(T_{k_1}(b_i)) T'_{k_1}(b_i)} \\
&\geq \sum_{\omega_N \in \{1,2\}^N} \sum_{i=1}^{2} \frac{p_{\omega_{N-1}}(T_{k_1}(b_i)) p_{k_1}(b_i) f(b_i)}{T'_{\omega_{N-1}}(T_{k_1}(b_i)) T'_{k_1}(b_i)} \\
&= \sum_{\omega_N \in \{1,2\}^N} \sum_{i=1}^{2} \frac{p_{\omega_{N-1}}(x_{N-1}) p_{k_1}(T^{-1}_{k_1,i} x_{N-1}) f(T^{-1}_{k_1,i} x_{N-1})}{T'_{\omega_{N-1}}(x_{N-1}) T'_{k_1}(T^{-1}_{k_1,i} x_{N-1})} \\
&= \sum_{\omega_{N-1} \in \{1,2\}^{N-1}} \sum_{k_1=1}^{2} \frac{p_{\omega_{N-1}}(x_{N-1})}{T'_{\omega_{N-1}}(x_{N-1})} \Big(\sum_{i=1}^{2} \frac{p_{k_1}(T^{-1}_{k_1,i} x_{N-1}) f(T^{-1}_{k_1,i} x_{N-1})}{T'_{k_1}(T^{-1}_{k_1,i} x_{N-1})}\Big) \\
&= \sum_{\omega_{N-1} \in \{1,2\}^{N-1}} \frac{p_{\omega_{N-1}}(x_{N-1,\omega_N})}{T'_{\omega_{N-1}}(x_{N-1,\omega_N})} \Big[\sum_{k_1=1}^{2} \sum_{i=1}^{2} \frac{p_{k_1}(T^{-1}_{k_1,i} x_{N-1,\omega_N}) f(T^{-1}_{k_1,i} x_{N-1,\omega_N})}{T'_{k_1}(T^{-1}_{k_1,i} x_{N-1,\omega_N})}\Big] \\
&= \sum_{\omega_{N-1} \in \{1,2\}^{N-1}} \frac{p_{\omega_{N-1}}(x_{N-1,\omega_N})}{T'_{\omega_{N-1}}(x_{N-1,\omega_N})} [\mathcal{L}_T f(x_{N-1,\omega_N})] \\
&\geq \sum_{\omega_{N-1} \in \{1,2\}^{N-1}} \frac{p_{\omega_{N-1}}(x_{N-1,\omega_N})}{T'_{\omega_{N-1}}(x_{N-1,\omega_N})} [\mathcal{L}_T f(\sigma)].
\end{aligned}$$

We have known that $\max\limits_{k \in \{1,2\}, x \in [0,\frac{1}{2}]} T'_k(x) = \frac{1}{2+\alpha}$ and $f(T^{-1}_{k,1}\sigma) > f(\sigma) \geq \frac{1}{2(1-\sigma)}$. Then, under the condition B: $\inf p_k(x) \geq \delta > 0$,

$$\begin{aligned}
\mathcal{L}_T f(\sigma) &= \frac{p_1(T^{-1}_{k,1}\sigma) f(T^{-1}_{k,1}\sigma)}{T'_k(T^{-1}_{k,1}\sigma)} + \frac{p_2(T^{-1}_{k,2}\sigma) f(T^{-1}_{k,2}\sigma)}{T'_k(T^{-1}_{k,2}\sigma)} \\
&\geq \frac{p_1(T^{-1}_{k,1}\sigma) f(T^{-1}_{k,1}\sigma)}{T'_k(T^{-1}_{k,1}\sigma)} \geq \frac{\delta}{2(1-\sigma)(2+\alpha)} > 0.
\end{aligned}$$

Therefore,

$$\mathcal{L}^N_T f(x) \quad \geq \quad \mathcal{L}_{T^N} f(1) \geq \gamma > 0,$$

where $\gamma = \frac{\delta}{2(1-\sigma)(2+\alpha)} \sum\limits_{\omega_{N-1} \in \{1,2\}^{N-1}} \frac{p_{\omega_{N-1}}(x_{N-1,\omega_N})}{T'_{\omega_{N-1}}(x_{N-1,\omega_N})}$ with N, σ depend only on A. Moreover, for $n > N$, we set $h(x) = \mathcal{L}^{n-N}_T f(x)$. Then $h(x) \in \mathcal{C}_A$,

$$\mathcal{L}^n_T f(x) \quad = \quad \mathcal{L}^N_T(\mathcal{L}^{n-N}_T f(x)) = \mathcal{L}^N_T h(x) \geq \gamma.$$

Thus, for all $n \geq N, \mathcal{L}^n_T f(x) \geq \gamma > 0$.

For last part, suppose that $\rho = \mathcal{L}_T \rho \in \mathcal{C}_A$. Clearly, if set E such that $m(E) = 0$, it follows that $\mu(E) = \int\limits_E \rho dm = 0$. Conversely, $\mu(E) = 0$. $\rho = \mathcal{L}^n_T \rho$ implies that $0 = \mu(E) = \int\limits_E \rho dm = \int\limits_E \mathcal{L}^n_T \rho dm \geq \gamma m(E)$. Hence, if $\rho = \mathcal{L}_T \rho$ then $\mu = \rho m$ is equivalent to m. □

In the following, we give the proof of Theorem 3.1

Proof. In Remark 3.5 we know the existence of ACIM for T. Next, we give the proof of uniqueness. Suppose random map T has two mutually singular ACIM μ_1 and μ_2. From Lemma 3.6, we have $I = A_k \subseteq \text{supp}(\mu_1)$ and $I = A_k \subseteq \text{supp}(\mu_2)$. Therefore, $I \subseteq \text{supp}(\mu_1) \cap \text{supp}(\mu_2)$. This contradicts the mutual singularity of μ_1 and μ_2. Thus, the random map T has a unique ACIM. From Proposition 3.7, the invariant density ρ is uniformly bounded below. ☐

4. EXAMPLE

We present an example of the random map T which satisfies the assumptions and then it preserves an ACIM.

Example 4.1. Let random map $T = \{T_1(x), T_2(x); p_1(x), p_2(x)\}$, for $0 < \beta < \alpha < 1$,

$$T_1 = \begin{cases} x(1 + 2^{\alpha}x^{\alpha}) & x \in [0, \frac{1}{2}), \\ 2x - 1 & x \in [\frac{1}{2}, 1]. \end{cases} \qquad T_2 = \begin{cases} x(1 + 2^{\beta}x^{\beta}) & x \in [0, \frac{1}{2}), \\ \frac{3}{2}x - \frac{3}{4} & x \in [\frac{1}{2}, 1]. \end{cases}$$

and

$$p_1 = \begin{cases} \frac{1+x^{\alpha}}{3} & x \in [0, \frac{1}{2}), \\ \frac{1}{3} & x \in [\frac{1}{2}, 1]. \end{cases} \qquad p_2 = \begin{cases} \frac{2-x^{\alpha}}{3} & x \in [0, \frac{1}{2}), \\ \frac{2}{3} & x \in [\frac{1}{2}, 1]. \end{cases} .$$

It's obvious that $p_1(x), p_2(x) \in [0, 1], p_1(x) + p_2(x) = 1, \forall x \in [0, 1]$ and $\inf_{x \in I} p_k(x) \geq \frac{1}{3} > 0$ satisfying condition (B). Then we will check if this random map satisfies condition (A), $\sum_{i=1}^{l} \frac{p_k(T_{k,i}^{-1}(x))}{T_k'(T_{k,i}^{-1}(x))}, 1 \leq l \leq 2$, is decreasing for all $k = 1, 2$. First, for $x \in [0, \frac{1}{2}), \frac{p_2(x)}{T_2'(x)}$ is decreasing directly from the given functions. But for $\frac{p_1(x)}{T_1'(x)}$ decreasing, we have to check if

$$p_1'(x)T_1'(x) - p_1(x)T_1''(x) \leq 0, \forall x \in [0, \frac{1}{2}).$$

Then for $x \in [\frac{1}{2}, 1], \frac{p_1(x)}{T_1'(x)} = \frac{1}{6}, \frac{p_2(x)}{T_2'(x)} = \frac{4}{9}$.

$$\begin{aligned} p_1'(x)T_1'(x) - p_1(x)T_1''(x) &= \frac{1}{3}\alpha x^{\alpha-1}[1 + (1+\alpha)2^{\alpha}x^{\alpha}] - \frac{1+x^{\alpha}}{3}[\alpha 2^{\alpha}(1+\alpha)x^{\alpha-1}] \\ &= \frac{\alpha x^{\alpha-1}}{3}[1 + (1+\alpha)2^{\alpha}x^{\alpha} - (1+x^{\alpha})2^{\alpha}(1+\alpha)] \\ &= \frac{\alpha x^{\alpha-1}}{3}[1 - 2^{\alpha}(1+\alpha)]. \end{aligned}$$

The term in square bracket is absolutely negative, i.e. $1 < 2^{\alpha}(1+\alpha), \forall \alpha \in (0, 1)$. So, $\frac{p_1(x)}{T_1'(x)}$ is decreasing from

$$\left(\frac{p_1(x)}{T_1'(x)}\right)' = \frac{p_1'(x)T_1'(x) - p_1(x)T_1''(x)}{(T_1'(x))^2} \leq 0.$$

Whence condition (A) is satisfied, since $x \mapsto T_{1,1}^{-1}x$ and $x \mapsto T_{2,1}^{-1}x$ are increasing. This random map preserves an ACIM.

5. STOCHASTIC STABILITY OF RANDOM MAP

In this section we study the random map $T_\varepsilon = \{T_1(x), T_{1,\varepsilon}(x); p_{1,\varepsilon}(x), p_{2,\varepsilon}(x)\}$, with $\varepsilon > 0, 0 < \alpha - \varepsilon < 1$

$$T_1 = \begin{cases} x(1+2^\alpha x^\alpha) & x \in [0, \frac{1}{2}), \\ g_1(x) & x \in [\frac{1}{2}, 1]. \end{cases} \qquad T_{1,\varepsilon} = \begin{cases} x(1+2^{\alpha-\varepsilon} x^{\alpha-\varepsilon}) & x \in [0, \frac{1}{2}), \\ g_{1,\varepsilon}(x) & x \in [\frac{1}{2}, 1]. \end{cases},$$

where $g_1'(x)$ and $g_{1,\varepsilon}'(x)$ are increasing functions with $g_1(\frac{1}{2}) = 0, g_{1,\varepsilon}(\frac{1}{2}) = 0$ and $g_1'(x) > 1, g_{1,\varepsilon}'(x) > 1$. Also, $p_{1,\varepsilon}(x) + p_{2,\varepsilon}(x) = 1$ and $0 \le p_{1,\varepsilon}(x), p_{2,\varepsilon}(x) \le 1$, for any $x \in I$. In addition to assumptions (A) and (B), we assume that
Assumption (C): $\lim_{\varepsilon\to 0} \sup_x p_{2,\varepsilon}(x) = 0$.

Let $\mathcal{L}_{T_\varepsilon}$ be Perron-Frobenius operator for random map T_ε. For each $0 < \varepsilon < \alpha$, we can treat $\beta = \alpha - \varepsilon$. From results of section 3, we know there exist a fixed point f_ε of $\mathcal{L}_{T_\varepsilon}$ and $f_\varepsilon \in \mathcal{C}_A$, for some $A \ge \frac{4}{1-\alpha}$. Since $\mathcal{C}_A$ is a convex and compact set, we can attain a subsequence $\{f_{\varepsilon_k}\}_{\varepsilon_k>0}$ of $\{f_\varepsilon\}_{\varepsilon>0}$ such that

$$f_{\varepsilon_k} \xrightarrow{L^1} f^* \in \mathcal{C}_A, \text{ as } k \to \infty, \quad \varepsilon_k \to 0.$$

Lemma 5.1. *Suppose* $f_{\varepsilon_k} \xrightarrow{L^1} f^* \in \mathcal{C}_A$. *Then* f^* *is a* T_1-*invariant density function, as* $\varepsilon \to 0$.

Proof. We know $\{f_{\varepsilon_k}\}_{\varepsilon_k>0}$ is a subsequence of $\{f_\varepsilon\}_{\varepsilon>0}$. Then we write f_{ε_k} as the fixed point of $\mathcal{L}_{T_{\varepsilon_k}}$, i.e. $f_{\varepsilon_k} = \mathcal{L}_{T_{\varepsilon_k}} f_{\varepsilon_k}$. Recall that P_{T_1} is the Perron-Frobenius operator for deterministic map T_1. Then,

$$\begin{aligned} \|f^* - P_{T_1} f^*\|_1 &\le \|f^* - f_{\varepsilon_k}\|_1 + \|f_{\varepsilon_k} - P_{T_1} f_{\varepsilon_k}\|_1 + \|P_{T_1} f_{\varepsilon_k} - P_{T_1} f^*\|_1 \\ &\le \|f^* - f_{\varepsilon_k}\|_1 + \|f_{\varepsilon_k} - P_{T_1} f_{\varepsilon_k}\|_1 + \|f_{\varepsilon_k} - f^*\|_1 \\ &= 2\|f^* - f_{\varepsilon_k}\|_1 + \|\mathcal{L}_{T_{\varepsilon_k}} f_{\varepsilon_k} - P_{T_1} f_{\varepsilon_k}\|_1. \end{aligned}$$

The first term on the right converges to 0 as $k \to \infty$ by the choice of subsequence. Moreover, we have

$$\begin{aligned} \|\mathcal{L}_{T_{\varepsilon_k}} f_{\varepsilon_k} - P_{T_1} f_{\varepsilon_k}\|_1 &= \|P_{T_1}(p_{1,\varepsilon_k} f_{\varepsilon_k}) + P_{T_{1,\varepsilon}}(p_{2,\varepsilon_k} f_{\varepsilon_k}) - P_{T_1} f_{\varepsilon_k}\|_1 \\ &= \|P_{T_1}(p_{1,\varepsilon_k} f_{\varepsilon_k} - f_{\varepsilon_k}) + P_{T_{1,\varepsilon}}(p_{2,\varepsilon_k} f_{\varepsilon_k})\|_1 \\ &= \|(P_{T_{1,\varepsilon}} - P_{T_1}) p_{2,\varepsilon_k} f_{\varepsilon_k}\|_1 \\ &\le 2\|p_{2,\varepsilon_k} f_{\varepsilon_k}\|_1 \le 2 \sup_x p_{2,\varepsilon_k} \to 0, \text{ as } \varepsilon_k \to 0. \end{aligned}$$

Thus, $f^* = P_{T_1} f^* \quad m-$a.e. □

Theorem 5.2. *Let* f_ε *be the unique invariant density of* T_ε. *Let* f^* *be the unique invariant density of* T_1. *Then,* $\lim_{\varepsilon\to 0} \|f_\varepsilon - f^*\|_1 = 0$.

Proof. By Lemma 5.1 there is a subsequence $\{f_{\varepsilon_k}\}_{\varepsilon_k>0}$ such that $f_{\varepsilon_k} \xrightarrow{L^1} f^* \in \mathcal{C}_A$ and $\mu = f^* m$ is an ACIM for T_1. By the uniqueness of ACIM for T_1, then all subsequences $\{f_{\varepsilon_{k_i}}\}_{\varepsilon_{k_i}>0}$ of $\{f_\varepsilon\}_{\varepsilon>0}$ have f^* as their common limit point. Hence, $\|f_\varepsilon - f^*\|_1 \to 0$, as $\varepsilon \to 0$. □

6. TWO DIMENSIONAL NON-UNIFORMLY EXPANDING MAP

In this section we show that our random maps give rise to an interesting family of 2-dimensional non-uniformly expanding maps which admit a unique ACIM.

Define the skew product $S(x,\omega) : I^2 \to I^2$ as in [3].

$$S(x,\omega) = (T_k(x), \varphi_{k,x}(\omega)), \text{ with } \begin{cases} \varphi_{1,x}(\omega) = \frac{\omega}{p_1(x)}, & \omega \in [0, p_1(x)), \\ \varphi_{2,x}(\omega) = \frac{\omega - p_1(x)}{p_2(x)}, & \omega \in [p_1(x), 1]. \end{cases}$$

In our problem there are four disjoint sets U_1, U_2, U_3, U_4 . Define $S_i = S\mid_{U_i}, i = 1, 2, 3, 4$.

$S_1 = (T_{1,1}(x), \varphi_{1,x}(\omega)) = (x(1+2^\alpha x^\alpha), \frac{\omega}{p_1(x)})$, $\quad U_1 = [0, \frac{1}{2}) \times [0, p_1(x))$,
$S_2 = (T_{1,2}(x), \varphi_{1,x}(\omega)) = (g_1(x), \frac{\omega}{p_1(x)})$, $\quad U_2 = [\frac{1}{2}, 1] \times [0, p_1(x))$,
$S_3 = (T_{2,2}(x), \varphi_{2,x}(\omega)) = (g_2(x), \frac{\omega - p_1(x)}{p_2(x)})$, $\quad U_3 = [\frac{1}{2}, 1] \times [p_1(x), 1]$,
$S_4 = (T_{2,1}(x), \varphi_{2,x}(\omega)) = (x(1+2^\beta x^\beta), \frac{\omega - p_1(x)}{p_2(x)})$, $\quad U_4 = [0, \frac{1}{2}) \times [p_1(x), 1]$.

We are interested in the Lyapunov exponents of skew product at the point $(0,0)$. Since skew product is a 2-dimensional map, we have two distinct Lyapunov exponents at each point. Define the Lyapunov exponents of skew product.

Definition 6.1. (**Lyapunov exponents**)

Let $S : I^2 \to I^2$ be a diffeomorphism on a manifold of dimension two. Let $|\cdot|$ be the norm on tangent vectors induced by a Riemannian metric on I^2. For each point $t \in I^2$ and vector v.

$$\lambda(t,v) = \limsup_{n\to\infty} \frac{1}{n} \log(|DS_t^n \cdot v|).$$

Focus on the point $(0,0)$, then the skew product is confirmed as $S = (T_{1,1}(x), \frac{\omega}{p_1(x)})$ and $S^n = (T_{1,1}^n(x), \frac{\omega}{p_1(x)\cdot\ldots\cdot p_1(T_{1,1}^n x)})$. Then, we have

$$DS^n = \begin{pmatrix} T_{1,1}^{'}(T_{1,1}^{n-1}x)\cdot \ldots \cdot T_{1,1}^{'}(x) & 0 \\ \frac{-\omega[p_1'(x)(p_1(T_{1,1}x)\cdot\ldots\cdot p_1(T_{1,1}^n x))+\ldots+p_1'(T_{1,1}^n x)\cdot(T_{1,1}^n x)'\cdot(p_1(x)\cdot\ldots\cdot p_1(T_{1,1}^{n-1}x))]}{[p_1(x)\cdot\ldots\cdot p_1(T_{1,1}^n x)]^2} & \frac{1}{p_1(x)\cdot\ldots\cdot{}_1(T_{1,1}^n x)} \end{pmatrix}$$

For convenient, we notate that $\lambda_1 = \lambda(t, v^1)$ and $\lambda_2 = \lambda(t, v^2)$ with

$$v^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad v^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and point } t = (0,0).$$

Also, we know that $T_{1,1}(0) = 0, T_{1,1}^{'}(0) = 1$ and $0 < p_1(0) < 1$. So,

$$\lambda_1 = \limsup_{n\to\infty} \frac{1}{n} \log(|DS_{(0,0)}^n \cdot v^1|) = \limsup_{n\to\infty} \frac{1}{n} \log(|\begin{pmatrix} 1 \\ 0 \end{pmatrix}|) = 0.$$

and

$$\lambda_1 = \limsup_{n\to\infty} \frac{1}{n} \log(|\begin{pmatrix} 0 \\ \frac{1}{p_1(0)} \end{pmatrix}|) = \log \frac{1}{p_1(0)} > 0.$$

Remark 6.2. These two exponents tell us that this map is non-uniformly expanding. Since our $\lambda_1 = 0$ at the point $(0,0)$, which implies that there is no expandsion or contraction in the v^1 direction at point $(0,0)$.

By our result on the random map (Theorem 3.1) and the result of [3], this skew product admits a unique ACIM of the form $\mu \times m$, where μ is the ACIM of the random map.

7. Appendix

Let $(I, \mathcal{B}(I), m)$ be a measure space, where $I = [0,1], \mathcal{B}(I)$ is Borel $\sigma-$algebra and m is Lebesgue measure. For $0 < \beta < \alpha < 1$, let

$$T = \begin{cases} x(1 + 2^{\alpha} x^{\alpha}) & x \in [0, \frac{1}{2}), \\ g(x) & x \in [\frac{1}{2}, 1]. \end{cases}$$

We assume:
(1) $g(\frac{1}{2}) = 0,$;
(2) $g^{'}(x) > 1$.

We study a deterministic map $T : I \to I$, with partition $\mathcal{P} = \{I_1, I_2\}, I_1 = [0, \frac{1}{2}], I_1 = [\frac{1}{2}, 1]$.

Lemma 7.1. *Let μ be a T ACIM. Then the support of μ is I.*

Proof. If $g(x)$ maps $[\frac{1}{2}, 1]$ into $[0,1]$, i.e. $g(x)$ is onto. The uniqueness of ACIM for T has been proved. We just only consider the other case, which $g(1) < 1$. For our map, it's obvious that $T[0, \frac{1}{2}] = [0,1]$. We need to show that for any interval $J \subset I$, there exists an $n \geq 1$ such that $T^n(J) \supseteq [0, \frac{1}{2}]$. If $J \supset I_k, T(J) \supseteq [0, \frac{1}{2}]$ obviously. Let $J \subset I_k$. Since $m(T(J)) > m(J)$, there exists a $j \geq 1$ such that $T^j(J)$ contains a partition point in its interior. We consider all three possible cases.

(1) If $T^j(J)$ contains the partition point 0, then obviously there exists a $k \geq 1$ such that $T^{j+k}(J) \supseteq [0, \frac{1}{2}]$.

(2) If $T^j(J)$ contains the partition point $\frac{1}{2}$ in its interior, i.e. $T^j(J) \supset (t_1, t_2)$ with $\frac{1}{2} \in (t_1, t_2)$. It's observed that $T[\frac{1}{2}, t_2] = [g(\frac{1}{2}), g(t_2)] = [0, g(t_2)]$, which contains the partition point 0. It becomes the first case.

(3) If $T^j(J)$ contains the partition point 1 in its interior. Let $T^j(J) \supset [s, 1]$, with $s > \frac{1}{2}$. Now, we first to show that $g(x) < x, x \in [\frac{1}{2}, 1]$. Let $\psi(x) = g(x) - x$. We have $\psi(1) = g(1) - 1 < 0$ since $g(x)$ is not onto. Also, $\psi'(x) = g'(x) - 1 > 0$, i.e. $\psi(x)$ is increasing for $x \in [\frac{1}{2}, 1]$. So, $\psi(x) < 0$ and then $g(x) < x, x \in [\frac{1}{2}, 1]$. Thus for some $k \geq 1, T^k[s, 1]$ must contain partition point $\frac{1}{2}$. Since $T[s, 1] = [g(s), g(1)]$, with $g(s) < s, g(1) < 1$ and $m[g(s), g(1)] > m[s, 1]$. That's, after iterations, the interval gets closer and closer to the point $\frac{1}{2}$ and the length of interval becomes longer and longer. There may be two possible cases. One is that $\frac{1}{2} \in [g^k(s), g^k(1)]$ with $[g^{k-1}(s), g^{k-1}(1)] \subseteq I_2$. The other is that for some $l, [g^{l-1}(s), g^{l-1}(1)] \subseteq I_2$, and $[g^l(s), g^l(1)] \subseteq I_1$. By the first branch of map, we know that for some $r, T^r[g^l(s), g^l(1)]$ must contain the point $\frac{1}{2}$.

Let A denote the support of μ. Since A contains an interval $J, T^n(J), n \geq 1$, and A is an invariant set. Consequently (by invariance) A must contain I. Moreover, $A \subset I$. Therefore, the support A is I. □

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